11 The Nyquist Stability Criterion

This chapter introduces a test for closed-loop stability, based only on the frequency response of the open-loop system. The method was developed by H. Nyquist in 1932, when he was studying stability problems of feedback amplifies in the Bell Labs. We first present an alternative method of displaying the frequency response of a system graphically: the Nyquist plot.

Nyquist Plots

In a Bode plot, the magnitude and phase of the response are shown in two separate diagrams. In a Nyquist plot, the frequency response is displayed in one diagram: the imaginary part is plotted versus the real part of $G(j\omega)$, and the frequency ω is taken as a parameter that varies from zero to infinity. Even though negative frequencies do not have a physical meaning it is sometimes useful to extend the frequency axis and let the parameter ω vary from $-\infty$ to $+\infty$.

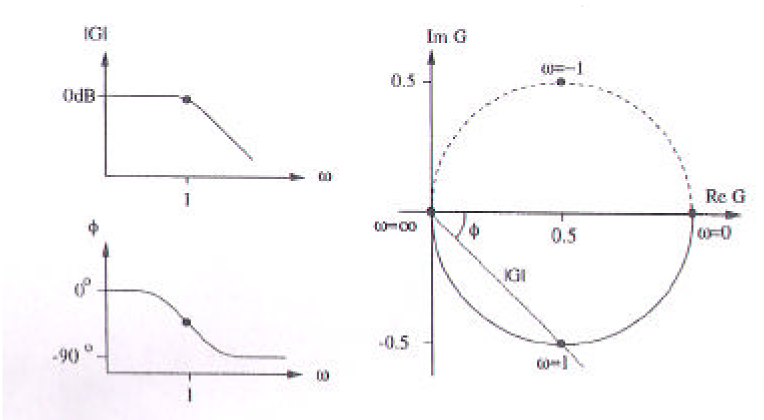

Figure 11.1: Bode and Nyquist plot for the first example.

To illustrate the idea of a Nyquist plot, we consider again the transfer function

$$G(s) = \frac{1}{s+1} \quad or \qquad G(jw) = \frac{1}{jw+1}$$

For ω=0, we have G=1, and for $\omega=\infty$, we have G = 0. At the corner frequency, ω=1, we have

$$G = \frac{1}{1+j} = \frac{1-j}{2} = \frac{1}{2} - j\frac{1}{2}$$

A sketch of the Bode plot of G(s) is shown in the left half of Fig. 11.1. The magnitude and phase taken by G(jω) at the frequencies zero and infinity can be read off the Bode plot as the limits of the magnitude and phase curves infinitely far to the left and right, respectively. The values of the magnitude are 1 (0 dB) and 0 (or -∞ dB) and the phase angles are 0° and –90°. At the corner frequency, the magnitude is $1/\sqrt{2}$ (-3 dB) and the phase is -45°. The complex values at the frequencies 0, 1 and ∞ are marked in the G(s)-plane in the right half of Fig. 11.1. In this way, magnitude and phase for any frequency can be read off the Bode diagram and plotted as a point in the G(s)-plane. Doing this for a sufficient number of frequencies reveals that the points are located on a semicircle centred at ½. This semicircular plot is the Nyquist plot of G(s) for positive frequencies.

Once the Nyquist plot for positive frequencies has been constructed, it can be extended to include negative frequencies by adding the mirror image across the real axis to the plot. This follows from

$$G(-jw) = \overline{G}(jw)$$

This part is shown as a dashed curve in Fig. 11.1. Hence, the complete Nyquist plot is a circle of radius 1/2 , centred at 1/2.

As a second example, consider the transfer function

$$G(s) = \frac{1}{s(s+1)}$$

Evaluated on the imaginary axis,

$$G(jw) = \frac{1}{jw(jw+1)} = \frac{1}{-w^2 + jw}$$
$$= \frac{-w^2 - jw}{w^4 + w^2} = -\frac{1}{w^2+1} - j\frac{1}{w(w^2+1)}$$

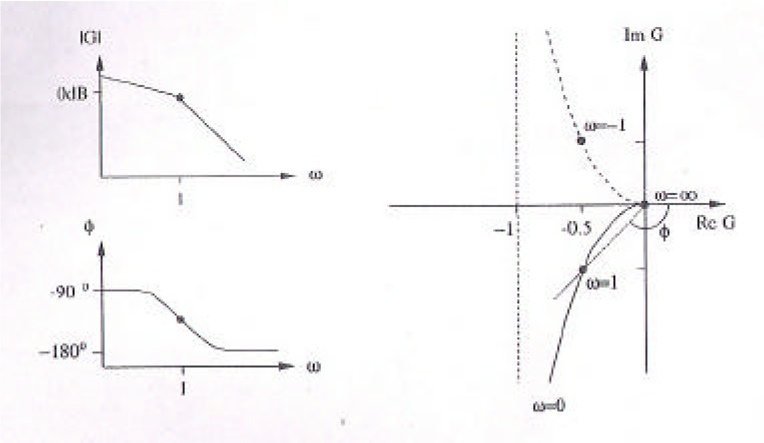

Figure 11.2: Bode and Nyquist plot for the second example

Hence, we have

ω=0: G=-1-j∞

ω=1: G=-0.5-j0.5

ω=∞: G=0

The Bode plots and the Nyquist diagram are shown in Fig. 11.2.

The Nyquist Stability Criterion

The Nyquist stability criterion is based on Cauchy's principle; the principle is to use the contour evaluation of an open-loop transfer function to determine the presence of unstable closed-loop poles.

Assume we are given a transfer function L(s)

$$L(s) = \frac{s-z}{(s-p_1)(s-p_2)}$$

The left side of Fig. 11.3 shows the s-plane, and the right side the L(s)-plane, into which the function L(s) maps its arguments (points of the s-plane). In the s-plane, two poles and a zero are marked, together with a test point s_0 which is moved clockwise along a closed contour C. We are interested in the change of the phase angle of $L(s_0)$, when s_0 is moved along the closed contour C. The phase angle of $L(s_0)$ is the sum of the zero angle and the negative pole angles indicated in the plot. It is clear that after a full traverse, the change of the phase angle is zero.

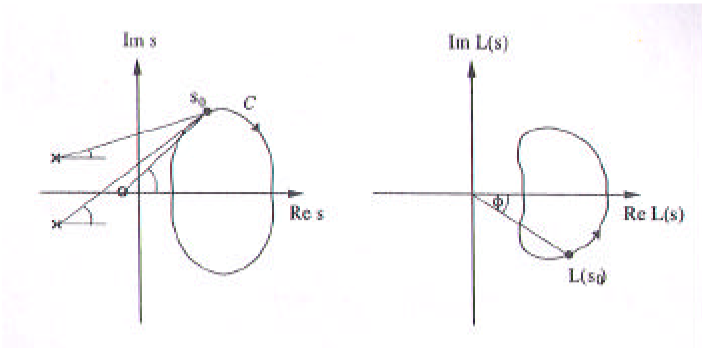

Figure 11.3: Contour evaluation

The function L(s) maps each point on C into the L(s)-plane, and the point $L(s_0)$ moves along a closed contour which is the mapping of C, as s_0 moves along C. Such a contour is shown in the right half of Fig. 11.3, where φ denotes the phase angle of $L(s_0)$. This plot also shows that the change of the phase angle of $L(s_0)$ after a full traverse is zero, i.e., $\Delta\varphi=0°$.

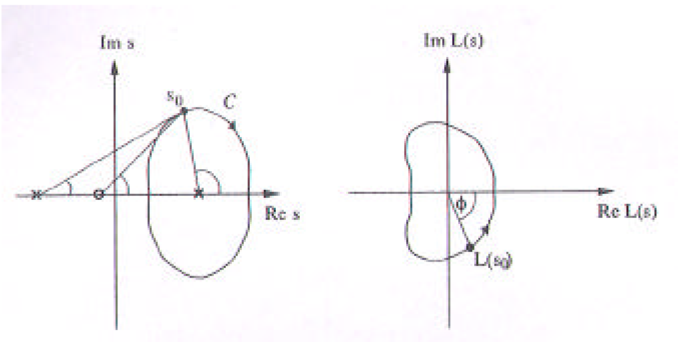

Figure 11.4: Contour evaluation with one pole enclosed

Fig. 11.4 shows a different example, where one of the poles of L(s) is enclosed by C. In this case, the phase change of $L(s_0)$ after a full traverse is not zero: the angle of the enclosed pole undergoes a change of 360°. If s_0 moves clockwise, then $L(s_0)$ moves counterclockwise and we have $\Delta\varphi=360°$. The closed contour into which C is mapped is shown in the right half of Fig. 11.4; note that this contour must encircle the origin of the L(s)-plane counterclockwise. If instead of a pole, a zero is enclosed, then $\Delta\varphi$=-360°, and L(s_0) moves clockwise.

In Fig. 11.5, two poles are enclosed and we have $\Delta\varphi$=720°. Therefore, the mapping of C must encircle the origin of the L(s)-plane twice counterclockwise.

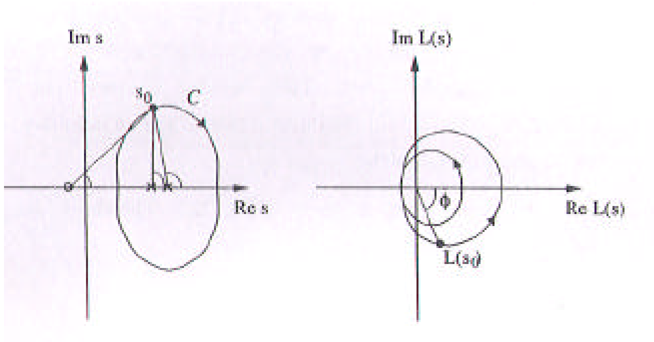

Figure 11.5: Contour evaluation with two poles enclosed

The above examples illustrate Cauchy's principle: the evaluation of L(s) along a closed contour C encircles the origin only if C encircles poles or zeros of L(s). Because C is oriented clockwise, an enclosed zero causes a clockwise encirclement of the origin, and an enclosed pole a counterclockwise encirclement.

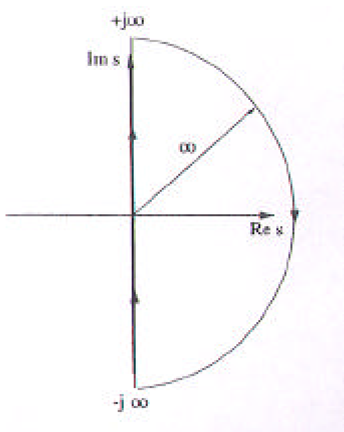

Figure 11.6: Nyquist path

Nyquist's concept was to use this fact to determine the presence of closed-loop poles in the right half plane. For this purpose, the contour C must be chosen such that it encloses the whole right half plane. This contour, known as the *Nyquist path*, is shown in Fig. 11.6. It consists of the imaginary axis and an infinitely large semicircle that encloses the right half plane.

Now consider the feedback system shown in Fig. 11.7. The above idea can be applied to check the closed-loop stability of this system.

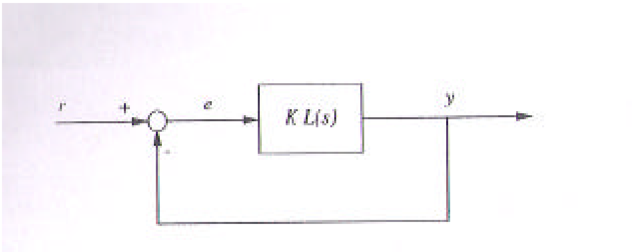

Figure 11.7: Closed-loop system

The closed-loop characteristic equation is

$$1+KL(s)=0$$

and an unstable zero of 1+KL(s) indicates an unstable closed-loop pole. We could evaluate this function along the Nyquist part, however we make the following simplification. Instead of evaluating 1+KL(s) and checking encirclements of the origin, we can equivalently evaluate the open-loop transfer function KL(s) and check encirclements of the point –1, see Fig. 11.8.

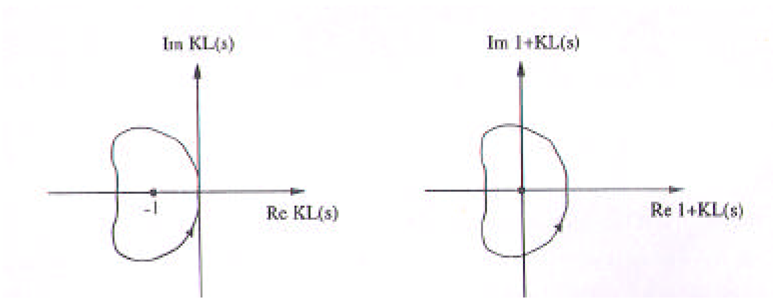

Figure 11.8: Shifting the evaluation of 1+KL(s) to the left

To evaluate KL(s) along the Nyquist path is equivalent to evaluating along the jω-axis from ω=-∞ to ω=+∞, and along the infinite arc. However, the transfer functions of physical systems are zero at infinite frequency, and the infinite arc in the s-plane is mapped to the origin of the L(s)-plane. Therefore, KL(s) needs to be evaluated only from -j∞ to +j∞, and that is precisely the Nyquist plot of KL(s) for positive and negative frequencies.

So far, we established the following:

> If the right half plane contains a zero or a pole of 1+KL(s), the Nyquist plot of KL(s) encircles the point –1 (clockwise or counterclockwise, respectively).

To distinguish between poles and zeros of the function 1+KL(s), we write L(s)=b(s)/a(s) to obtain

$$1+KL(s)=\frac{a(s)+Kb(s)}{a(s)}$$

This shows that

- the zeros of the function 1+KL(s) are the closed-loop poles, and
- the poles of the function 1+KL(s) are the open-loop poles (the poles of L(s)).

We assume that if the open loop transfer function is unstable, the number of unstable poles is known, so that these can be taken into account when checking closed-loop stability. If we let Z denote the number of unstable closed-loop poles, and P the number of unstable open-loop poles, then the above considerations show that the Nyquist plot of KL(s) encircles N=Z-P times clockwise the point –1 (a negative value for N indicates counterclockwise encirclements). Closed-loop stability can now be checked as follows:

a) Draw the Nyquist plot of KL(s);
b) Determine the number N of clockwise encirclements of point –1;
c) The closed-loop system has Z=N+P unstable poles.

If the open loop transfer function is stable, then the closed loop system is stable if the Nyquist plot does not encircle point –1. If KL(s) has one unstable pole, then closed-loop stability requires one counterclockwise encirclement of –1, etc.

Example 1

Consider

$$KL(s) = \frac{K}{s+1}$$

The Nyquist plot is shown in Fig. 11.9. It is a circle centred at K/2 with radius K/2. Because P=0 (no unstable poles of L(s)) and N=0 (no encirclements of –1) for all K>0, the closed-loop system is stable for all positive K.

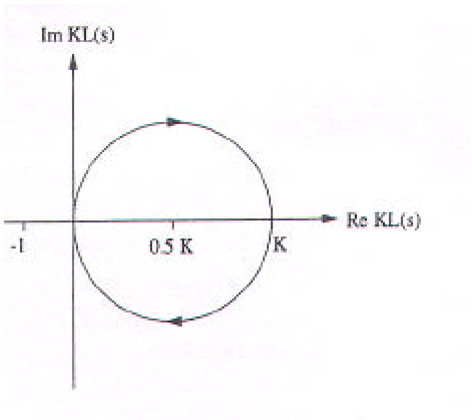

Figure 11.9: Nyquist stability test for the first example

Example 2

The Nyquist plot of the open-loop transfer function

$$KL(s) = \frac{K}{s(s+1)}$$

is obtained by scaling the Nyquist plot of Fig. 11.2 by the factor K. Apparently, this Nyquist plot is not a closed contour, even though it is the mapping of the Nyquist path, which is a closed contour. This is due to the fact that the Nyquist path as shown in Fig. 11.6 passes through a pole of L(s) at the origin. At a pole, the value of the transfer function is infinite, and the Nyquist plot is in fact closed by an arc at infinity. To see why this is so, consider the path shown in Fig. 11.10. The Nyquist path has been modified to avoid the pole at the origin by making an infinitely small detour to the right. It is this small arc that is mapped into an arc at infinity, and to determine whether or not the Nyquist plot encircles the point –1, it is important to know whether this infinite arc encloses the right half plane. On the small arc in Fig. 11.10, three test points s_i, i=1, 2, 3 are marked. Because the radius of the semicircle around the origin is infinitely small, the magnitude of $KL(s_i)$ is infinite and the phase angles- determined by the pole angles- are

$$\arg KL(s_1)=+90°, \qquad \arg KL(s_2)=0°, \qquad \arg KL(s_3)=-90°$$

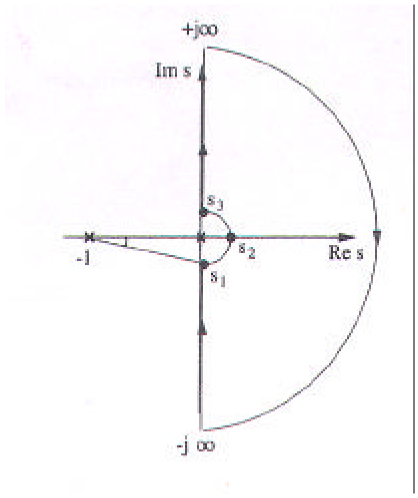

Figure 11.10: Modified Nyquist path

This shows that the Nyquist plot is completed by an infinite arc to the right, as shown in Fig. 11.11. Because the modified Nyquist path does not encircle the pole at the origin, we have P=0, and from Fig. 11.11, we conclude that there is no encirclement of point –1 for any positive value of K (i.e., N=0), and therefore, the closed-loop system is stable for any positive gain.

Example 3

Consider the transfer function

$$L(s) = \frac{10}{(s+1)^3} = \frac{10}{s^3 + 3s^2 + 3s + 1}$$

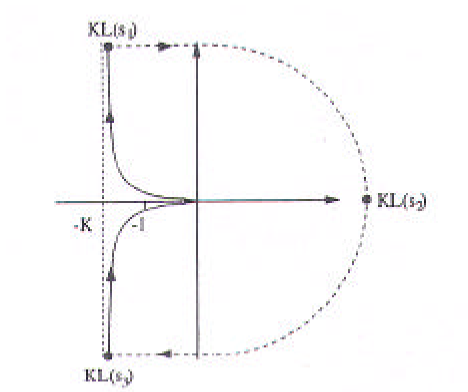

Figure 11.11: Nyquist stability test for the second example

Substituting jω for s gives

$$L(jw) = \frac{10}{(jw)^3 + 3(jw)^2 + 3jw + 1}$$

At low frequencies, ω<<1, we have

$$L \approx 10 \Rightarrow \quad |L| \approx 10, f \approx 0°$$

and at high frequencies

$$L \approx \frac{10}{(jw)^3} = \frac{10}{-jw^3} = j\frac{10}{w^3} \Rightarrow \quad |L| \approx \frac{10}{w^3}, f \approx -270°$$

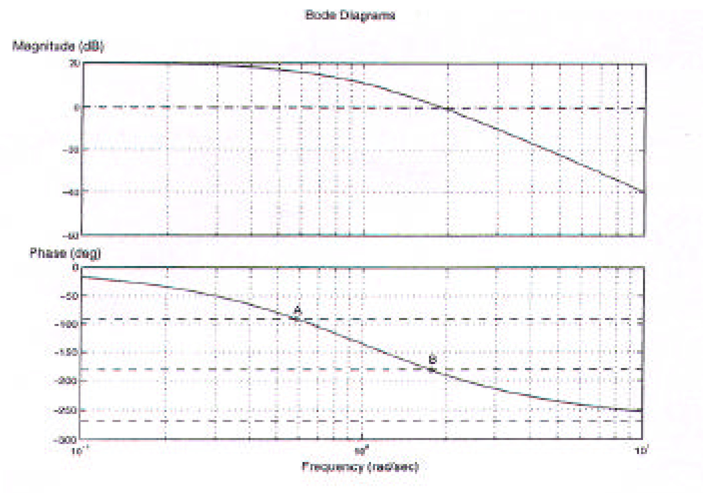

Figure 11.12: Bode plot of L(s)

The Bode plot of this function is shown in Fig. 11.12. The points marked A and B in the Bode plot correspond to the points, where the Nyquist plot intersects the negative imaginary and the negative real axis, respectively. The phase angle of -270° that can be read off the Bode plot for high frequencies (where the magnitude approaches zero) indicates that the Nyquist plot approaches the origin from above. The Nyquist plot of L(s) (which is the same as the Nyquist plot of KL(s) when K=1) encircles the critical point –1 twice clockwise (N=2). Because L(s) has no unstable poles (P=0), this implies that the closed-loop system has two unstable poles when K=1.

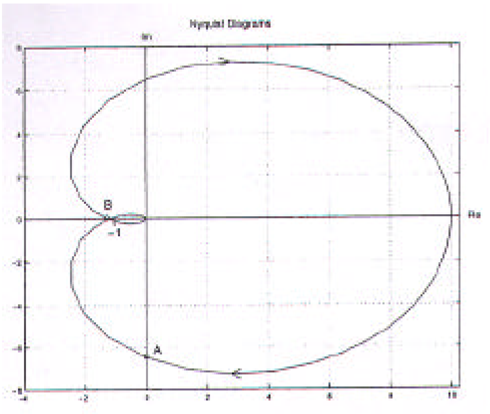

Figure 11.13: Nyquist plot of L(s)

Fig. 11.14 shows the Nyquist plot with K=0.5. The Nyquist plot of L(s) has been 'scaled down' to an extent that the critical point –1 is not encircled any longer. With this gain, the closed-loop system is stable.

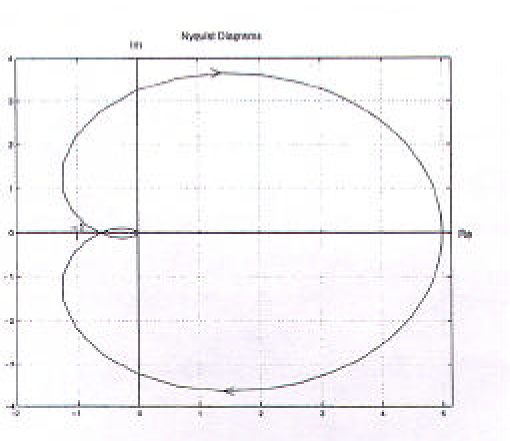

Figure 11.14: Nyquist plot of KL(s), K=0.5

To determine closed-loop stability for different values of K, it is convenient to use the Nyquist plot of L(s) instead of KL(s): the Nyquiest plot of KL(s) encircles the point –1 precisely when the Nyquist plot of L(s) encircles the point –1/K. Therefore, closed-loop stability can be assessed by counting the encirclements of the point –1/K by the Nyquiest plot of L(s). Fig. 11.15 shows that the Nyquist plot of L(s) crosses the negative real axis at –1.25, therefore the Nyquist plot of KL(s) does not encircle the critical point –1 when the gain is less than 0.8, and the closed-loop system is stable for all 0<K<0.8.

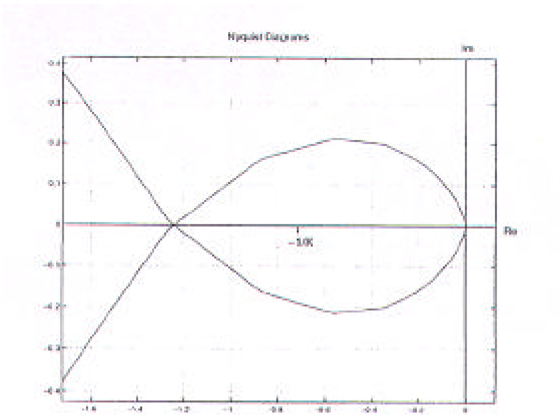

Figure 11.15: Nyquist plot of L(s)